Classification of Symplectic Automorphism Groups of Smooth Cubic Fourfolds

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Background

• Three standard ways to construct and compactify moduli spaces: GIT (Geometric Invariant Theory), Hodge theory (Baily-Borel, Toroidal compactifications), Minimal Model Program (KSBA compactifications).

• I look at objects whose moduli have GIT models and can be well approached via Hodge theory.

• All of those objects up to my knowledge are closely related to abelian varieties, $K3$ surfaces, cubic fourfolds and hyper-Kähler manifolds. Reason: some versions of global Torelli hold for those objects.

• In this talk we concentrate on cubic fourfolds. With Radu Laza, we completely classified the symplectic automorphism groups of smooth cubic fourfolds. We also gave geometric realizations for many of these groups.

Our Objects

Throughout this talk we consider complex algebraic varieties.

• A $K3$ surface is a simply-connected smooth compact surface with an everywhere non-degenerate holomorphic two form.

• A cubic fourfold is a degree 3 hypersurface in $\mathbb{P}^5$.

• A hyper-Kähler manifold is a simply-connected compact Kähler manifold Y with $H^0(\Omega^*_Y) = \mathbb{C}[\sigma]$, where σ is an everywhere non-degenerate holomorphic two form.

Proposition

1. *For smooth cubic fourfold, we have* $h^{3,1} = h^{1,3} = 1$, $h^{2,2} = 21$.
2. *For smooth* $K3$ *surface, we have* $h^{2,0} = h^{0,2} = 1$, $h^{1,1} = 20$.

Hyper-Kähler Manifolds

All known HK manifolds:

1. Hilbert schemes of points on $K3$ surfaces;
2. Kernels of $A^{[n+1]} \longrightarrow A$, where A is a two dimensional complex tori;
3. O'Grady 6 and O'Grady 10 (found by O'Grady, around 2000);
4. Their deformations.

Theorem (Beauville-Donagi, 1985)

The Fano scheme of lines on a smooth cubic fourfold is a hyper-Kähler fourfold of deformation type $K3^{[2]}$.

• More relations between cubic fourfolds and $K3$/HK manifolds were investigated from both perspectives of Hodge theory (Hassett, 2000) and derived category (Kuznetsov, 2010).

Theorem (global Torelli for $K3$)

For any two $K3$ surfaces S_1, S_2. If there exists $\varphi\colon H^2(S_1,\mathbb{Z}) \cong H^2(S_2,\mathbb{Z})$ an Hodge isomotry, such that $\varphi(\mathcal{K}_{S_1}) \cap \mathcal{K}_{S_2} \neq \emptyset$, then there exists uniquely an isomorphism $f\colon S_2 \cong S_1$, such that $f^* = \varphi$.

• Mathieu group M_{24} is the subgroup of S_{24} (acting naturally on $\mathbb{F}_2^{24}$) preserving the Golay code. Mathieu group M_{23} is the stabilizer of a coordinate of $\mathbb{F}_2^{24}$. They are sporadic simple. The order of M_{24} is 244823040.

Theorem (Mukai, 1988)

A finite group G is a subgroup of the symplectic automorphism group $\mathrm{Aut}^s(S)$ of a $K3$ surface if and only if G is a subgroup of M_{23} such that the induced action of G on $\{1,\ldots,24\}$ has at least 5 orbits.

Theorem (Voisin-Hassett-Laza-Looijenga)

Let $\mathcal{M}$ be the moduli space of smooth cubic fourfolds. Let $\mathcal{D}$ and Γ be the associated period domain (for the polarized VHS on middle cohomology), and monodromy group respectively. Then the period map gives an isomorphism of quasi-projective varieties

$$\mathcal{P}\colon \mathcal{M} \xrightarrow{\sim} \Gamma\backslash(\mathcal{D} - \mathcal{H}_2 - \mathcal{H}_6)$$

where $\mathcal{H}_2$ and $\mathcal{H}_6$ are the hyperplane arrangements associated to the short and long roots in $H^4_0(X,\mathbb{Z}) \cong A_2 \oplus E_8^2 \oplus U^2$.

Remark

Here $\mathcal{C}_2 = \Gamma\backslash\mathcal{H}_2$ and $\mathcal{C}_6 = \Gamma\backslash\mathcal{H}_6$ are the degree 2 and 6 divisors defined by Hassett, and can be realized as moduli spaces of degree 2 and degree 6 $K3$ surfaces respectively.

A strong version of global Torelli theorem for cubic fourfold holds:

Theorem

The period map $\mathscr{P}$ identifies the orbifold structures on $\mathcal{M}$ and its image in $\mathcal{D}/\Gamma$. In particular, for any smooth cubic fourfold X, we have $\mathrm{Aut}(X) \cong \mathrm{Aut}(H^4(X,\mathbb{Z}), H^2, HS)$.

Definition

An automorphism g of a smooth cubic fourfold X is called symplectic if its induced action on $H^{3,1}(X)$ is trivial. We denote by $\mathrm{Aut}^s(X)$ the group of all symplectic automorphisms of X.

Main theorem

Theorem (Laza-Zheng, 2019)

Let X be a smooth cubic fourfold with symplectic automorphism group $G = \mathrm{Aut}^s(X)$. Let $\mathcal{F}$ be the moduli space of cubic fourfold with the specified symplectic action by group G. Then we have and only have the following situations (geometric realizations are also obtained in many cases):

- Case $\dim(\mathcal{F}) = 20,\ G = 1$.
- Case $\dim(\mathcal{F}) = 12,\ G = 2$.
- Case $\dim(\mathcal{F}) = 8,\ G = 2^2$ or $G = 3$.
- Case $\dim(\mathcal{F}) = 6,\ G = S_3$ or 4.
- Case $\dim(\mathcal{F}) = 5,\ G = D_8$.
- Case $\dim(\mathcal{F}) = 4,\ G = A_{3,3}, D_{12}, A_4, D_{10}$.
- Case $\dim(\mathcal{F}) = 3,\ G = S_4$ or Q_8.
- Case $\dim(\mathcal{F}) = 2$,
  $G = 3^{1+4} : 2, A_{4,3}, A_5, 3^2.4, S_{3,3}, F_{21}, Hol(5)$ or QD_{16}.
- Case $\dim(\mathcal{F}) = 1$,
  $G = 3^{1+4} : 2.2, A_6, \mathrm{PSL}(2, \mathbb{F}_7), S_5, M_9, 3^2.D_8$ or T_{48}.
- Case $\dim(\mathcal{F}) = 0$,
  $G = 3^4 : A_6, A_7, 3^{1+4} : 2.2^2, M_{10}, \mathrm{PSL}(2, \mathbb{F}_{11})$ or $(3 \times A_5) : 2$.

Theorem (Yu-Zheng, 2018)

Let X be a smooth cubic fourfold with a faithful action of a finite group G. We can use method in GIT to construct a moduli space $\mathcal{F}$ for smooth cubic fourfolds with the same type of action by G. This space $\mathcal{F}$ is an irreducible normal quasi-projective variety. We have a Hermitian symmetric domain $\mathbb{D}$ (which is either a ball or type IV domain) with a properly discontinuously action by an arithmetic group Γ, such that there exists a period map $\mathscr{P}: \mathcal{F} \longrightarrow \Gamma\backslash\mathbb{D}$, which is an algebraic open embedding. Moreover, there is an extension: $\mathscr{P}: \overline{\mathcal{F}} \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$, where $\overline{\mathcal{F}}$ is the GIT-compactification of $\mathcal{F}$, $\mathcal{H}_*$ is a Γ-invariant hyperplane arrangement in $\mathbb{D}$, and $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ is the Looijenga compactification of $\Gamma\backslash(\mathbb{D} - \mathcal{H}_*)$.

Theorem (Laza-Zheng)

Let (G, S) a Leech pair such that $\mathrm{rank}(S) = 20$ and there exists a smooth cubic fourfold X with $G = \mathrm{Aut}^s(X)$ and $(G, S) \cong (G, S_G(X))$. We denote by T the orthogonal complement of S in $H_0^4(X, \mathbb{Z})$. Then we have and only have the following possibilities:

(1) *$G = 3^4 : A_6$, the corresponding cubic fourfold is the Fermat one $X(3^4 : A_6) = V(x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 + x_6^3)$ and $T = -(6^3 6) = A_2(-3)$. Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order 9. It holds $\mathrm{Aut}(X)/\mathrm{Aut}^s(X) \cong \mathbb{Z}/6$.*

Theorem

(2) $G = A_7$, *there are two smooth cubic fourfolds with symplectic action of* G. *One of them is* $X^1(A_7) = V(x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3-(x_1+x_2+x_3+x_4+x_5+x_6)^3)$ *and* $T = -(2^1 18)$. *It holds* $\mathrm{Aut}(X^1)/\mathrm{Aut}^s(X^1) \cong \mathbb{Z}/2$. *The other one* $X^2(A_7)$ *has* $T = -(18^3 18)$ *and admits no non-symplectic automorphisms.*

(3) $G = 3^{1+4} : 2.2^2$, *the cubic fourfold is* $X(3^{1+4} : 2.2^2) = V(x_1^3+x_2^3+x_3^3+x_4^3+x_5^3+x_6^3-3(\sqrt{3}+1)(x_1x_2x_3+x_4x_5x_6))$ *and* $T = -(6^0 6) = (A_1 \oplus A_1)(-3)$. *Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order* 12. *It holds* $\mathrm{Aut}(X)/\mathrm{Aut}^s(X) \cong \mathbb{Z}/4$.

Theorem

(4) *$G = M_{10}$, there are two smooth cubic fourfolds with symplectic action of G, and both of them have $T = -(12^{0}30)$.*

(5) *$G = L_2(11)$, the cubic fourfold is $X(L_2(11)) = V(x_1^3 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_2)$ and $T = -(22^{11}22) = A_2(-11)$. Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order 11. It holds $\mathrm{Aut}(X)/\mathrm{Aut}^s(X) \cong \mathbb{Z}/3$.*

(6) *$G = (3 \times A_5) : 2$, the cubic fourfold is $X((3 \times A_5) : 2) = V(x_1^3 + x_2^3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_6 + x_6^2x_3)$ and $T = -(10^{5}10) = A_2(-5)$. Moreover, this is the only smooth cubic fourfold with a symplectic automorphism of order 15. It holds $\mathrm{Aut}(X)/\mathrm{Aut}^s(X) \cong \mathbb{Z}/6$.*

Definition

There is a unique positive unimodular even lattice of rank 24 with no roots, denoted by $\mathbb{L}$, called the Leech lattice. The automorphism group of $\mathbb{L}$ is the Conway group Co_0. The group $\mathrm{Co}_1 = \mathrm{Co}_0/\{\pm 1\}$ is a sporadic simple group of order $4,157,776,806,543,360,000$.

Definition

A pair (G, S) consisting of a finite group G acting faithfully and point-wise freely on a positive definite even lattice S is called a Leech pair if:

(i) There is no root in S;

(ii) The induced action of G on the discriminant group $A_S = S^*/S$ is trivial.

Theorem (Gaberdiel-Hohenegger-Volpato, 2012)

Let (G, S) be a Leech pair with $\mathrm{rank}(S) + l(A_S) \leq 24$. Then there exists a primitive embedding of S into $\mathbb{L}$. As a corollary, the group G is a subgroup of Co_0.

Let X be a smooth cubic fourfold with $G < \mathrm{Aut}^s(X)$. Denote by $S = S_G(X)$ the covariant sublattice of $H^4(X, \mathbb{Z})$ under the induced action of G. Then $(G, S_G(X))$ is a Leech pair with $\mathrm{rank}(S) + l(A_S) \leq 23$. Thus $(G, S_G(X))$ is a subpair of $(\mathrm{Co}_0, \mathbb{L})$.

Theorem (Höhn-Mason, 2016)

There is a classification for saturated Leech pairs inside $(\mathrm{Co}_0, \mathbb{L})$. There are 290 up to conjugate. The discriminant form q_S for each (G, S) is also given.

Theorem (Laza-Zheng, 2019)

For a Leech pair (G, S), the following three statements are equivalent:

(i) *there exists a smooth cubic fourfold X with G acting faithfully and symplectically, such that $(G, S_G(X)) \cong (G, S)$;*

(ii) *there exists a faithful action of G on the Leech lattice $\mathbb{L}$ with $K = \mathbb{L}^G$ the invariant sublattice and $S_G(\mathbb{L}) = K^\perp$ the covariant sublattice, such that $(G, S_G(\mathbb{L})) \cong (G, S)$ and there exists an embedding of E_6 into $K \oplus U^2$;*

(iii) *there exists an embedding of $S \oplus E_6$ into the Borcherds lattice $\mathbb{B} = E_8^3 \oplus U^2$, such that the image of S is primitive.*

Strategy of the proof for the classification

- A smooth cubic fourfold X with $G = \mathrm{Aut}^s(X)$ induces a Leech pair $(G, S_G(X))$ inside $(\mathrm{Co}_0, \mathbb{L})$. What we need to do is to single out from the 290 Leech pairs in Höhn-Mason classification the ones rising from cubic fourfolds.
- Now given a Leech pair (G, S) inside $(\mathrm{Co}_0, \mathbb{L})$. We check whether this satisfies the third condition in the criterion. There are two possibilities: $S \oplus E_6$ has a saturation or not. Denote by $\widetilde{S}$ the lattice $S \oplus E_6$ or its saturation.
- The existence of a primitive embedding of $\widetilde{S}$ into $\mathbb{B}$ is equivalent to the existence of a signature $(20 - \mathrm{rank}(S), 2)$ even lattice with discriminant form equal to $-q_{\widetilde{S}}$. The latter can be checked by Nikulin's existence theorem for even lattices.
- Same group may have different geometric realizations. Such questions can be also reduced to lattice theory, and were answered in our paper.

Idea to obtain normal forms for a given symplectic automorphism group

• Suppose A is an abelian group appearing as a subgroup in a possible symplectic automorphism group (one of the 34). Then there exist projective representations of A on $\mathbb{P}^5$, preserving some smooth cubic fourfolds.

• By explicit understanding such representations, one is able to give normal forms for those invariant cubic fourfolds.

• The GIT moduli for these invariant cubic fourfolds is well-understood by a work of Yu-Zheng. Assume the dimension is n. Then there exists a Leech pair (G, S), with G in the 34-list, and $\mathrm{rank}(S) = 20 - n$.

• Suppose there is only one such Leech pair (this happens in many cases), then we conclude that a smooth cubic fourfold with such an action by A must have symplectic automorphism group G.

Thank You!